JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 22, No. 2, March–April 1999

Model Reduction with a Finite Horizon H_∞ Criterion

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An important problem in flight control and flying qualities is the approximation of a complex high-order system by a low-order model. For a given reduced-order model, we define the correlation measure between the plant and the model outputs to be the minimum of the ratio of weighted signal energy to weighted error energy. We give a criterion for the evaluation of the correlation measure in terms of minimization of a parameter occurring in a two-point boundary-value problem. Once the correlation measure for a given reduced-order model can be evaluated, a nonlinear programming algorithm can be used to select a model that maximizes the correlation between the plant and model outputs. The correlation index used can be regarded as an extension of the H_∞ performance criterion to the finite interval time-varying case. However, the usual H_∞ problem seeks an optimal controller, whereas our problem is to select the reduced-order model matrices that give the best correlation index. We also give an expression for the variation of the correlation due to parameter variations. The utilization of the theory is demonstrated by means of some examples.

Nomenclature

q = pitch rate
v = perturbed airspeed
α = angle of attack
δ_a = elevon control
δ_c = canard control
δ_e = elevator control
δ_t = throttle control
θ = pitch angle
ξ_1 = first fuselage bending mode
ξ_2 = second fuselage bending mode

I. Introduction

MODEL reduction is an important problem in the case of airplanes with significant aeroservoelastic dynamics. The original model in such cases is of high order, and thus, the resulting controller will have a complex structure, especially if it uses full state feedback. Also for highly augmented aircraft with flight and propulsive controls, it is useful to develop low-order models to analyze flying qualities.

If the aim is to design a low-order controller for a high-order plant, there are at least three broad approaches to achieve this. A general account of these three approaches is given in Ref. 1. The so-called direct design methods assume a stabilizing controller of fixed degree and seek to find the controller that maximizes a quadratic performance index.[2]

Another approach is to get a high-order controller by some design technique, such as linear quadratic Gaussian or H_∞, and then to approximate the high-order controller by a low-order one that possesses certain desirable properties. The third approach is to approximate the high-order plant by a low-order one. Then a low-order controller is designed and used to control the original plant. In this paper we concentrate on this approach and consider the problem of approximating the original plant by a low-order model in an optimal sense.

There are several model reduction techniques based on balanced realization of the state-space system of a transfer function. In these methods we can have an estimate of the infinity norm of the error between the reduced transfer function and the plant transfer function. Reference 3 initiated the work on balanced realizations, Ref. 4 extended the work for frequency weighting, and Ref. 5 gives an alternate computational method.

We present a model reduction technique based on an H_∞ time-domain criterion. It has the advantage of comparing directly the output of the reduced-order model with that of the plant utilizing a minimax criterion. Preliminary results in this direction were reported in Refs. 6 and 7. We also give criteria that need to be maximized at the optimal reduced-order model. At the optimal reduced-order model, the correlation measure will be maximized. We also give expressions for the derivatives of the correlation measure with respect to the reduced-order model parameters. Also a new numerical method for the evaluation of the correlation measure will be presented. The new method will be utilized in the computation of the optimal reduced-order model in a number of cases.

We now state the main problem. For the sake of generality, we pose it for time-varying systems. Let the plant be described by

$$\dot{\boldsymbol{x}}_p = A_p(t)\boldsymbol{x}_p + B_p(t)\boldsymbol{u}, \qquad \boldsymbol{x}_p(t_0) = 0 \tag{1}$$

$$\boldsymbol{y}_p = C_p(t)\boldsymbol{x}_p + D_p(t)\boldsymbol{u} \tag{2}$$

where $\boldsymbol{x}_p(t)$, $\boldsymbol{u}(t)$, and $\boldsymbol{y}_p(t)$ are the plant state vector, the control vector, and the plant output vector, respectively. Let the reduced-order model that approximates the plant be chosen to be

$$\dot{\boldsymbol{x}}_m = A_m(t)\boldsymbol{x}_m + B_m(t)\boldsymbol{u}, \qquad \boldsymbol{x}_m(t_0) = 0 \tag{3}$$

$$\boldsymbol{y}_m = C_m(t)\boldsymbol{x}_m + D_m(t)\boldsymbol{u} \tag{4}$$

where $\boldsymbol{x}_m(t)$ and $\boldsymbol{y}_m(t)$ are, respectively, the state vector and the output vector of the reduced-order model.

For given $A_m(t)$, $B_m(t)$, $C_m(t)$, and $D_m(t)$, let $\boldsymbol{u}$ be chosen such that the correlation index given by

$$\frac{\int_{t_0}^{T} \frac{1}{2}\boldsymbol{u}^*(t)R(t)\boldsymbol{u}(t)\,\mathrm{d}t}{\int_{t_0}^{T} \frac{1}{2}(\boldsymbol{y}_p - \boldsymbol{y}_m)^*Q(t)(\boldsymbol{y}_p - \boldsymbol{y}_m)\,\mathrm{d}t} \tag{5}$$

is minimized. The superscript * denotes matrix or vector transpose. Let this minimum value be denoted by ρ. Thus, $\boldsymbol{u}$ represents the worst input, and ρ gives a measure of the worst-case correlation between the plant output and the model output. The problem is to choose $A_m(t)$, $B_m(t)$, $C_m(t)$, and $D_m(t)$ such that ρ is maximized.

Received Dec. 1, 1997; revision received Sept. 25, 1998; accepted for publication Sept. 28, 1998.

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The problem just stated is essentially the minimax optimization problem

$$\min_{u} \max_{A_m, B_m, C_m, D_m} \frac{\int_{t_0}^{T} \frac{1}{2}\boldsymbol{u}^*(t)R(t)\boldsymbol{u}(t)\,\mathrm{d}t}{\int_{t_0}^{T} \frac{1}{2}(\boldsymbol{y}_p - \boldsymbol{y}_m)^* Q(t)(\boldsymbol{y}_p - \boldsymbol{y}_m)\,\mathrm{d}t}$$

Conceptually it represents the problem of finding the model matrices A_m, B_m, C_m, and D_m so that the model output $\boldsymbol{y}_m$ is closest (in the L_2 sense) to the plant output $\boldsymbol{y}_p$ in the presence of the worst possible input. This problem is significantly different from conventional system identification methods that use input–output matching. In the present setup, the model identification is done under the worst possible input. In the time-invariant case, the model matrices A_m, B_m, C_m, and D_m are selected so that the norm of the transfer function from the input $\boldsymbol{u}$ to the error $\boldsymbol{y}_p - \boldsymbol{y}_m$ is minimized.

Because Eq. (5) represents the ratio of weighted signal energy to weighted error energy, the problem may be regarded as a modified H_∞ problem except for a few differences. We consider time-varying systems, and in our case the interval of control is finite. There are extensions of the H_∞ results to the finite horizon time-varying case.[8] However, our approach is different and is based on considering a minimax problem and the associated two-point boundary-value problem. Also, the general aim of H_∞ problems is the design of a suboptimal controller, whereas in this paper we are interested in the selection of reduced model matrices. It is necessary in our case to use nonlinear programming algorithms to select the model matrices that maximize ρ.

In the case of time-invariant systems, a nonlinear programming algorithm can be used to find at least a local maximum of ρ. For the time-varying case, the matrices $A_m(t)$, $B_m(t)$, $C_m(t)$, and $D_m(t)$ need to be expressed in terms of basis functions, and a nonlinear programming algorithm needs to be used to maximize ρ with respect to the coefficients of the basis functions.

We do not require the plant and the model to be open-loop stable. This is significant because many of the modern aircraft have open-loop unstable poles. We show in Sec. IV by means of examples that the method is indeed applicable to such cases. There is yet another advantage of our method. One of the criticisms in the approach of getting a low-order model from a high-order plant is that satisfactory approximation of the plant requires some knowledge in advance of the controller.[1] Because we maximize the correlation between the plant and model outputs for the worst-case input, the correlation in the case of any other controller is bound to be better. Thus, our method furnishes a satisfactory approximation without requiring an a priori knowledge of the controller.

We now give a summary of the results of the paper. In Sec. II, conditions that characterize the worst input are derived for a given model. A two-point boundary-value problem needs to be solved for the least positive ρ to obtain the worst-case correlation between the outputs of the plant and the model. In Sec. IV we present an algorithm for the evaluation of ρ. A nonlinear programming algorithm can then be used to find the model matrices that maximize ρ.

In Sec. III we derive an expression for the variation of correlation between the plant and model outputs as a functional of the variations in system parameters. Making use of this result, we derive expressions for the derivatives of ρ with respect to the elements of A_m, B_m, and C_m. These expressions are useful in case the nonlinear programming algorithm makes use of derivatives. From the expressions for the derivatives, conditions that need to be satisfied at the optimal reduced-order model can be derived.

In Sec. IV some examples are worked out. Finally, certain conclusions are given in Sec. V.

II. Computation of ρ for a Given Reduced-Order Model

Assume that the matrices $A_m(t)$, $B_m(t)$, $C_m(t)$, and $D_m(t)$ are given certain values, which are not necessarily the optimal values. In this section, we characterize ρ as the minimum positive value for which a certain two-point boundary-value problem has a nontrivial solution. Also, we derive a computationally useful characterization of ρ.

Let

$$\boldsymbol{x} = \begin{pmatrix} \boldsymbol{x}_p^* & \boldsymbol{x}_m^* \end{pmatrix}^* \tag{6}$$

$$\boldsymbol{y} = \boldsymbol{y}_p - \boldsymbol{y}_m \tag{7}$$

$$A(t) = \begin{pmatrix} A_p & 0 \\ 0 & A_m \end{pmatrix} \tag{8}$$

$$B(t) = \begin{pmatrix} B_p \\ B_m \end{pmatrix} \tag{9}$$

$$C(t) = (C_p \quad -C_m) \tag{10}$$

$$D(t) = D_p - D_m \tag{11}$$

Assuming that $D_m(t) = D_p(t)$, we can write Eqs. (1–4) as

$$\dot{\boldsymbol{x}} = A(t)\boldsymbol{x} + B(t)\boldsymbol{u}, \qquad \boldsymbol{x}(t_0) = 0 \tag{12}$$

$$\boldsymbol{y} = C(t)\boldsymbol{x} \tag{13}$$

The performance index given by Eq. (5) can be put in the form

$$J(u) = \frac{\int_{t_0}^{T} \frac{1}{2}\boldsymbol{u}^*(t)R(t)\boldsymbol{u}(t)\,\mathrm{d}t}{\int_{t_0}^{T} \frac{1}{2}\boldsymbol{x}^*(t)W(t)\boldsymbol{x}(t)\,\mathrm{d}t} \tag{14}$$

where

$$W(t) = C^*(t)Q(t)C(t) \tag{15}$$

The problem is to characterize $\boldsymbol{u}(t)$ that minimizes the index in Eq. (14).

Let $\rho = \inf_u J(u)$. We assume that for all t

$$R(t) > 0, \qquad Q(t) \geq 0 \tag{16}$$

Equation (16) guarantees that the numerator and denominator of Eq. (14) are nonnegative for any $\boldsymbol{u}$. The necessary conditions that characterize the worst input can be stated as follows.

Theorem 1. Consider the system given by Eqs. (12–15). If $\boldsymbol{u}(t)$ minimizes the correlation index given by Eq. (14), then there exists an adjoint vector $\psi(t)$, not identically zero, such that

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = -A^*\psi - \rho W\boldsymbol{x}, \qquad \psi(T) = 0 \tag{17}$$

with

$$\boldsymbol{u}(t) = R^{-1}B^*\psi \tag{18}$$

Proof. For a proof, see Theorem 4.1, Chapter 3, of Ref. 7.

Let

$$\hat{A} = A \tag{19}$$

$$\hat{B} = BR^{-1}B^* \tag{20}$$

$$\hat{C} = -\rho W \tag{21}$$

Thus, we have a two-point boundary-value problem given by

$$\begin{pmatrix} \dot{\boldsymbol{x}} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & -\hat{A}^* \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \psi \end{pmatrix} \tag{22}$$

with

$$\boldsymbol{x}(t_0) = 0, \qquad \psi(T) = 0 \tag{23}$$

The following theorem follows from Theorem 4.2, Chapter 3, of Ref. 8.

Theorem 2. Let (x, ψ) satisfy the boundary-value problem given by Eqs. (22) and (23) for the least positive ρ such that

$$\int_{t_0}^{T} \frac{1}{2}\boldsymbol{x}^* W\boldsymbol{x}\,\mathrm{d}t > 0$$

where $u = R^{-1}B^*\psi$. Then ρ is the minimum value of the index given by Eq. (14) and $\boldsymbol{u}$ is the worst input.

To find the least positive value ρ that yields a nontrivial solution for Eqs. (22) and (23), a computational technique is given in Chapter 3 of Ref. 8 that utilizes the transition matrix associated with Eq. (22). The details of this method will be given in Sec. IV.

III. Derivatives and Optimality Conditions

In this section we derive an expression for the variation in the correlation measure ρ in terms of variations in the system matrices. Making use of this expression, for the time-invariant case, we can derive expressions for the derivative of ρ with respect to the elements of the reduced-order matrices. We can further derive the conditions that need to be satisfied at the optimal reduced-order model.

Consider Eqs. (1–10). The system equations are given by

$$\dot{\boldsymbol{x}} = A(t)\boldsymbol{x} + B(t)\boldsymbol{u}, \qquad \boldsymbol{x}(t_0) = 0 \tag{24}$$

$$\boldsymbol{y} = C(t)\boldsymbol{x} \tag{25}$$

We can write Eq. (5) as

$$\frac{\int_{t_0}^{T} \frac{1}{2}\boldsymbol{u}^*(t)R(t)\boldsymbol{u}(t)\,\mathrm{d}t}{\int_{t_0}^{T} \frac{1}{2}\boldsymbol{x}^*(t)C^*(t)Q(t)C(t)\boldsymbol{x}(t)\,\mathrm{d}t} \tag{26}$$

For given $A_m(t)$, $B_m(t)$, and $C_m(t)$, let ρ be the minimum of the correlation index in Eq. (26) over $\boldsymbol{u}(t)$. Let the elemental variations in $A_m(t)$, $B_m(t)$, and $C_m(t)$ be denoted by $\delta A_m(t)$, $\delta B_m(t)$, and $\delta C_m(t)$, respectively. Let $\delta A(t)$, $\delta B(t)$, and $\delta C(t)$ be the variations in the matrices $A(t)$, $B(t)$, and $C(t)$ corresponding to the elemental variations $\delta A_m(t)$, $\delta B_m(t)$, and $\delta C_m(t)$. Notice that

$$\delta A(t) = \begin{pmatrix} 0 & 0 \\ 0 & \delta A_m(t) \end{pmatrix} \tag{27}$$

$$\delta B(t) = \begin{pmatrix} 0 \\ \delta B_m(t) \end{pmatrix} \tag{28}$$

$$\delta C(t) = (0 \quad \delta C_m(t)) \tag{29}$$

Let $\delta\rho$ denote the variation in ρ caused by δA, δB, and δC. We now derive an expression for $\delta\rho$ in terms of $\delta A(t)$, $\delta B(t)$, and $\delta C(t)$. For given $A_m(t)$, $B_m(t)$, and $C_m(t)$, let $\boldsymbol{u}$ minimize the index in Eq. (26). In the following, we suppress the dependence of the matrices on t for simplicity of notation. From Eqs. (19–23), we get the following boundary-value problem, which needs to be satisfied by the corresponding pair (x, ψ):

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + BR^{-1}B^*\psi \tag{30}$$

$$\dot{\psi} = -\rho C^*QC\boldsymbol{x} - A^*\psi \tag{31}$$

$$\boldsymbol{x}(t_0) = \psi(T) = 0 \tag{32}$$

Let $\boldsymbol{x}_1$ and ψ_1 represent the variations in $\boldsymbol{x}$ and ψ due to δA, δB, and δC. From Eqs. (30–32), we have the following equations satisfied by $\boldsymbol{x}_1$ and ψ_1:

$$\dot{\boldsymbol{x}}_1 = A\boldsymbol{x}_1 + \delta A\boldsymbol{x} + BR^{-1}B^*\psi_1 + (BR^{-1}\delta B^* + \delta BR^{-1}B^*)\psi \tag{33}$$

$$\dot{\psi}_1 = -\delta\rho C^*QC\boldsymbol{x} - \rho(\delta C^*QC + C^*Q\delta C)\boldsymbol{x} - \rho C^*QC\boldsymbol{x}_1 - A^*\psi_1 - \delta A^*\psi \tag{34}$$

$$\boldsymbol{x}_1(t_0) = \psi_1(T) = 0 \tag{35}$$

Theorem 3. Consider Eqs. (30–35). Let

$$\Omega = -\int_{t_0}^{T} \psi^*\delta Ax\,\mathrm{d}t - \int_{t_0}^{T} \psi^*B^*R^{-1}\delta B\psi\,\mathrm{d}t - \rho\int_{t_0}^{T} \boldsymbol{x}^*C^*Q\delta C\boldsymbol{x}\,\mathrm{d}t \tag{36}$$

Then the variation in ρ is given by

$$\delta\rho = \frac{\Omega}{\int_{t_0}^{T} \frac{1}{2}\boldsymbol{x}^*C^*QC\boldsymbol{x}\,\mathrm{d}t} \tag{37}$$

Proof. From Eq. (34), we get

$$\int_{t_0}^{T} \boldsymbol{x}^*\dot{\psi}_1\,\mathrm{d}t = -\delta\rho\int_{t_0}^{T} \boldsymbol{x}^*C^*QC\boldsymbol{x}\,\mathrm{d}t - \rho\int_{t_0}^{T} \boldsymbol{x}^*(\delta C^*QC + C^*Q\delta C)\boldsymbol{x}\,\mathrm{d}t - \rho\int_{t_0}^{T} \boldsymbol{x}^*C^*QC\boldsymbol{x}_1\,\mathrm{d}t - \int_{t_0}^{T} \boldsymbol{x}^*A^*\psi_1\,\mathrm{d}t - \int_{t_0}^{T} \boldsymbol{x}^*\delta A^*\psi\,\mathrm{d}t \tag{38}$$

Also, by an integration by parts and by Eqs. (30), (32), and (35),

$$\int_{t_0}^{T} \boldsymbol{x}\dot{\psi}_1\,\mathrm{d}t = -\int_{t_0}^{T} \boldsymbol{x}^*A\psi_1\,\mathrm{d}t - \int_{t_0}^{T} \psi^*BR^{-1}B^*\psi_1\,\mathrm{d}t \tag{39}$$

Because

$$\int_{t_0}^{T} \boldsymbol{x}^*(\delta C^*QC + C^*Q\delta C)\boldsymbol{x}\,\mathrm{d}t = 2\int_{t_0}^{T} \boldsymbol{x}^*C^*Q\delta C\boldsymbol{x}\,\mathrm{d}t \tag{40}$$

from Eqs. (38) and (39), we get

$$\delta\rho\int_{t_0}^{T} \boldsymbol{x}^*C^*QC\boldsymbol{x}\,\mathrm{d}t + 2\rho\int_{t_0}^{T} \boldsymbol{x}^*C^*Q\delta C\boldsymbol{x}\,\mathrm{d}t + \rho\int_{t_0}^{T} \boldsymbol{x}^*C^*QC\boldsymbol{x}_1\,\mathrm{d}t + \int_{t_0}^{T} \boldsymbol{x}^*\delta A^*\psi\,\mathrm{d}t = \int_{t_0}^{T} \psi^*BR^{-1}B^*\psi_1\,\mathrm{d}t \tag{41}$$

From Eq. (31),

$$\rho\int_{t_0}^{T} \boldsymbol{x}^*C^*QC\boldsymbol{x}_1\,\mathrm{d}t = -\int_{t_0}^{T} (\dot{\psi} + A^*\psi)^*\boldsymbol{x}_1\,\mathrm{d}t \tag{42}$$

Integrating the first term of the integrand from the right side of Eq. (42) by parts, and using Eqs. (32) and (35), we get

$$\rho\int_{t_0}^{T} \boldsymbol{x}^*C^*QC\boldsymbol{x}_1\,\mathrm{d}t = \int_{t_0}^{T} \psi^*\delta A\boldsymbol{x}\,\mathrm{d}t + \int_{t_0}^{T} \psi^*BR^{-1}B^*\psi_1\,\mathrm{d}t + \int_{t_0}^{T} \psi^*(BR^{-1}\delta B^* + \delta BR^{-1}B^*)\psi\,\mathrm{d}t \tag{43}$$

Incorporating Eq. (43) in Eq. (41) and using the fact that

$$\int_{t_0}^{T} \psi^*(BR^{-1}\delta B^* + \delta BR^{-1}B^*)\psi\,\mathrm{d}t = 2\int_{t_0}^{T} \psi^*BR^{-1}\delta B\psi\,\mathrm{d}t \tag{44}$$

we get Eq. (37). □

Using Eq. (37), the variation in the correlation measure ρ due to parameter variations can be computed for any given $A_m(t)$, $B_m(t)$, and $C_m(t)$. Incorporating Eqs. (27–29) in Eq. (37), we get

$$\delta\rho = \frac{-\int_{t_0}^{T}\psi_m^*\delta A_m \boldsymbol{x}_m\,\mathrm{d}t - \int_{t_0}^{T}\psi^* BR^{-1}\delta B_m^*\psi_m\,\mathrm{d}t + \rho\int_{t_0}^{T}\boldsymbol{x}^*C^*Q\delta C_m\boldsymbol{x}_m\,\mathrm{d}t}{\int_{t_0}^{T}\frac{1}{2}x^*C^*QC\boldsymbol{x}\,\mathrm{d}t} \tag{45}$$

For the time-invariant case, from Eq. (45), we get

$$\frac{\partial\rho}{\partial A_m} = \frac{-\int_{t_0}^{T}\psi_m\boldsymbol{x}_m^*\,\mathrm{d}t}{\int_{t_0}^{T}\frac{1}{2}\boldsymbol{x}^*C^*QC\boldsymbol{x}\,\mathrm{d}t} \tag{46}$$

$$\frac{\partial\rho}{\partial B_m} = \frac{-\int_{t_0}^{T}\psi_m\psi^*BR^{-1}\,\mathrm{d}t}{\int_{t_0}^{T}\frac{1}{2}\boldsymbol{x}^*C^*QC\boldsymbol{x}\,\mathrm{d}t} \tag{47}$$

$$\frac{\partial\rho}{\partial C_m} = \frac{\rho\int_{t_0}^{T}QC\boldsymbol{x}\boldsymbol{x}_m^*\,\mathrm{d}t}{\int_{t_0}^{T}\frac{1}{2}\boldsymbol{x}^*C^*QC\boldsymbol{x}\,\mathrm{d}t} \tag{48}$$

Thus, at the optimal A_m, B_m, and C_m, we have

$$\int_{t_0}^{T}\psi_m\boldsymbol{x}_m^*\,\mathrm{d}t = 0 \tag{49}$$

$$\int_{t_0}^{T}\psi_m\psi^*BR^{-1}\,\mathrm{d}t = 0 \tag{50}$$

$$\int_{t_0}^{T}QC\boldsymbol{x}\boldsymbol{x}_m^*\,\mathrm{d}t = 0 \tag{51}$$

Equations (46–48) represent the gradient of the correlation index ρ with respect to the model parameters A_m, B_m, and C_m. In the next section we shall use these coefficients to compute the optimal reduced-order model.

IV. Numerical Computation of ρ

As discussed in Sec. II, the model reduction problem can be expressed as a minimax optimization problem of the performance criterion

$$\sup_{A_m,B_m,C_m}\inf_{u} J(A_m, B_m, C_m, \boldsymbol{u}) \tag{52}$$

where

$$J = \frac{\int_{t_0}^{T}\frac{1}{2}\boldsymbol{u}^*(t)R(t)\boldsymbol{u}(t)\,\mathrm{d}t}{\int_{t_0}^{T}\frac{1}{2}\boldsymbol{x}^*(t)W(t)\boldsymbol{x}(t)\,\mathrm{d}t} \tag{53}$$

As shown in Theorem 1, for a given set of model parameters, minimization of $J(\cdot, \boldsymbol{u})$ with respect to the control $\boldsymbol{u}$ leads to the necessary condition that the two-point boundary-value problem given by Eqs. (22) and (23) be satisfied for the least value of the parameter ρ, where $\rho = \inf_u J(\boldsymbol{u})$. The second stage of the minimax problem is then maximization of ρ with respect to the model parameters A_m, B_m, and C_m. This is equivalent to the following problem:

$$\sup_{A_m,B_m,C_m}\rho(A_m, B_m, C_m) \tag{54}$$

subject to the differential constraint given by Eqs. (22) and (23).

From a numerical perspective, the problem is, therefore, solved in two steps: 1) find $\rho = \inf_u J(\cdot, \boldsymbol{u})$, i.e., minimization of J for a given set of model matrices and 2) maximization of ρ with respect to the model parameters. In what follows, we describe the details for solving these two steps. For this purpose we assume that the system is time invariant so that the plant matrices as well as the model matrices are constant. In addition, we assume that the weighting matrices in the cost function in Eq. (14) are also constant.

Minimization of $J(\cdot, \boldsymbol{u})$ for Given Set of Model Matrices

For a given set of model parameters, the two-point boundary-value problem given by Eqs. (22) and (23) has nontrivial solutions for countably many values of the parameter ρ. Here we seek the smallest value of $\rho > 0$ for which the boundary-value problem has a nontrivial solution.

Let $\Phi(t, \tau)$ be the state transition matrix associated with Eq. (22). Then we have

$$\begin{bmatrix} x(T) \\ \psi(T) \end{bmatrix} = \Phi\left(T, \frac{T+t_0}{2}\right)\Phi\left(\frac{T+t_0}{2}, t_0\right)\begin{bmatrix} \boldsymbol{x}(t_0) \\ \boldsymbol{\psi}(t_0) \end{bmatrix} \tag{55}$$

Premultiplying Eq. (55) by $\Phi^{-1}[T, (T+t_0)/2]$ and denoting

$$\Phi\left(\frac{T+t_0}{2}, t_0\right) = \begin{pmatrix} \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \end{pmatrix}$$

$$\Phi^{-1}\left(T, \frac{T+t_0}{2}\right) = \begin{pmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{21} & \zeta_{22} \end{pmatrix}$$

we obtain

$$\begin{pmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{21} & \zeta_{22} \end{pmatrix}\begin{bmatrix} \boldsymbol{x}(T) \\ \boldsymbol{\psi}(T) \end{bmatrix} = \begin{pmatrix} \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \end{pmatrix}\begin{bmatrix} \boldsymbol{x}(t_0) \\ \boldsymbol{\psi}(t_0) \end{bmatrix} \tag{56}$$

Using the boundary conditions in Eq. (23), this gives

$$\begin{pmatrix} \zeta_{11} & \nu_{12} \\ \zeta_{21} & \nu_{22} \end{pmatrix}\begin{bmatrix} \boldsymbol{x}(T) \\ \boldsymbol{\psi}(t_0) \end{bmatrix} = 0$$

Clearly, a nontrivial solution for ρ is obtained if the determinant of coefficient matrix is zero, i.e.,

$$\det\begin{pmatrix} \zeta_{11} & \nu_{12} \\ \zeta_{21} & \nu_{22} \end{pmatrix} = 0 \tag{57}$$

The problem of finding the smallest ρ then reduces to a line search over a finite range of positive values of ρ and finding the value of ρ at which the determinant changes sign. Because this involves only a one-dimensional line search, a simple computation of the determinant of the coefficient matrix in Eq. (57) quickly identifies the smallest ρ for a nontrivial solution of Eqs. (22) and (23). This completes the first part of the numerical computation.

Maximization of ρ over Model Matrices

For maximization of ρ over the space of model matrices, we employ a gradient approach. This is essentially a maximization problem of a scalar cost function over a multidimensional parameter space. To proceed with our method, at the kth iteration we choose a set of model matrices that are not necessarily optimal, which we denote as A_m^k, B_m^k, and C_m^k. Then for maximization the $(k+1)$th iterate, the model matrices are taken as

$$A_m^{k+1} = A_m^k + \varepsilon\frac{\partial\rho^k}{\partial A_m^k}, \qquad B_m^{k+1} = B_m^k + \varepsilon\frac{\partial\rho^k}{\partial B_m^k}$$
$$C_m^{k+1} = C_m^k + \varepsilon\frac{\partial\rho^k}{\partial C_m^k} \tag{58}$$

where the parameter $\varepsilon > 0$ must be chosen small enough so that $\rho^{k+1} > \rho^k$. Note that the gradient of the correlation index with respect to the model matrix A_m is given by $\partial\rho^k/\partial A_m^k$. Similarly, $\partial\rho^k/\partial B_m^k$ and $\partial\rho^k/\partial C_m^k$ define the gradient with respect to the model matrices B_m and C_m, respectively. These gradients were obtained in Sec. III and are given by Eqs. (46), (47), and (48), respectively.

The optimal value of the parameter ρ is obtained by repeating the iteration in Eq. (58) until convergence, or until $|\rho^{i+1} - \rho^i| < \delta$, where δ is a small positive number.

For the proof of convergence of the given algorithm, we note that by the Taylor's expansion,

$$\rho^{k+1} = \rho^k + \left\langle\left\langle \frac{\partial\rho^k}{\partial A_m^k}, \left(A_m^{k+1} - A_m^k\right)\right\rangle\right\rangle + \left\langle\left\langle \frac{\partial\rho^k}{\partial B_m^k}, \left(B_m^{k+1} - B_m^k\right)\right\rangle\right\rangle + \left\langle\left\langle \frac{\partial\rho^k}{\partial C_m^k}, \left(C_m^{k+1} - C_m^k\right)\right\rangle\right\rangle + \text{higher-order terms}$$

where we have used the notation

$$\langle\langle M, N\rangle\rangle = \sum_i \sum_j M_{ij} N_{ij}$$

Clearly if we use the update law in Eq. (58), we have

$$\rho^{k+1} = \rho^k + \varepsilon\left\{\left\|\frac{\partial\rho^k}{\partial A_m^k}\right\|^2 + \left\|\frac{\partial\rho^i}{\partial B_m^k}\right\|^2 + \left\|\frac{\partial\rho^k}{\partial C_m^k}\right\|^2\right\} + \text{higher-order terms}$$

$$\left\|\frac{\partial\rho}{\partial M}\right\|^2 = \sum_{ij}\left|\frac{\partial\rho}{\partial M_{ij}}\right|^2$$

This shows that $\rho^{i+1} \geq \rho^i$ provided ε is chosen sufficiently small. This completes the proof of convergence.

The preceding maximization procedure utilizes the gradient of the cost function, and as in any gradient-based algorithm, the method actually converges to the local minimum nearest to the point of initialization. One approach to overcome the problem of convergence to a local minimum is to vary the initial guess of the matrices and repeat the computation. Additionally, the model matrices can be given large perturbations near the local minimum so as to force the iteration move to other regions of the parameter space of model matrices.

The method is based on finite horizon output matching only and provides a reduced-order model that produces a good approximation of the system output within this time horizon. In this respect the presence of unstable modes in the reduced-order model is not of much of a concern because it is expected that the reduced-order model will be used for control system design for the corresponding finite time horizon only.

Example 1

For illustration consider the following example taken from Ref. 9. The plant matrices are given by

$$A_p = \begin{pmatrix} -3 & -4 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad B_p = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$C_p = (0 \quad 0 \quad 2)$$

It is desired to obtain a second-order model that has similar time-response characteristics for the output $y(t)$ over a time interval of 5 s. The weighting matrices were taken as $Q = 1000$ and $R = 1$. Following the method described in the preceding section, we assume a model

$$A_{\text{assumed}} = \begin{pmatrix} -2.8 & -1.8 \\ 1.0 & 0.0 \end{pmatrix}, \qquad B_{\text{assumed}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$C_{\text{assumed}} = (0 \quad 1)$$

The iterative method described earlier converged to the reduced-order model

$$A_m = \begin{pmatrix} -2.0168 & -1.8771 \\ 1.1170 & 0.6387 \end{pmatrix}, \qquad B_m = \begin{pmatrix} 0.9686 \\ -0.1593 \end{pmatrix}$$

$$C_m = (-0.0507 \quad 1.0159)$$

The values of the model correlation index ρ are 0.0063 for the assumed model and 0.4863 for the optimal model.

A reduced-order model has been derived in Ref. 9 based on the frequency response criterion that the following relation be satisfied as closely as possible:

$$\frac{|G_H(j\omega)|^2}{|G_L(j\omega)|^2} = 1, \qquad 0 \leq \omega \leq \infty$$

where G_H and G_L are the high-order and low-order system transfer functions, respectively. This reduced-order model[9] is given by

$$A_{[9]} = \begin{pmatrix} -2.828 & -2.0 \\ 1.0 & 0.0 \end{pmatrix}, \qquad B_{[9]} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad C_{[9]} = (0 \quad 2)$$

Figure 1 compares the response trajectories for the system output corresponding to a unit step input. As shown in Fig. 1 the response of the H_∞ reduced-order model is very close to the response of the actual system.

Example 2: Aircraft Model with Structural Modes

We consider the model reduction problem for an advanced supersonic transport.[10] The longitudinal airframe of the aircraft with two fuselage bending modes is described by an eighth-order model of the form

$$\dot{\boldsymbol{x}}_p = A_p \boldsymbol{x}_p + B_p \boldsymbol{u}, \qquad \boldsymbol{x}_p(0) = 0, \qquad \boldsymbol{y}_p = C_p \boldsymbol{x}_p \tag{59}$$

where the various state and control variables are

$$\boldsymbol{x}_p = (v, \alpha, \theta, q, \xi_1, \dot{\xi}_1, \xi_2, \dot{\xi}_2)^*, \qquad \boldsymbol{u} = (\delta_e, \delta_t, \delta_c, \delta_a)^*$$

The units for the airspeed are feet per second, and the angles and the control surface deflections are in degrees. The various matrices in the plant model given by Eq. (59) under flight condition of supersonic cruise at Mach 2.5 are as follows:

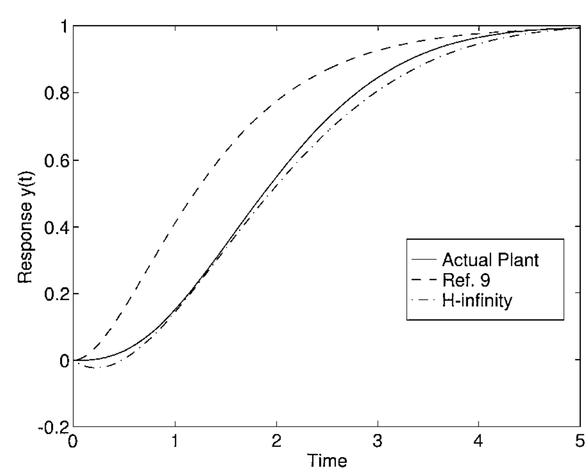

Fig. 1 Comparison of responses for a unit step input.

$$A_p = \begin{pmatrix} -0.0127 & -0.0136 & -0.0360 & 0 & 0 & 0 & 0 & 0 \\ -0.0969 & -0.4010 & 0 & 0.9610 & 19.5900 & -0.1185 & -9.2000 & -0.1326 \\ 0 & 0 & 0 & 1.0000 & 0 & 0 & 0 & 0 \\ -0.2290 & 1.7260 & 0 & -0.7220 & -12.0200 & -0.3420 & 1.8420 & 0.8810 \\ 0 & 0 & 0 & 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0.1204 & 0 & 0.0496 & -44.0000 & -1.2740 & -4.0300 & -0.5080 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0000 \\ 0 & 0.1473 & 0 & 0.3010 & -7.4900 & -0.1257 & -21.7000 & -0.8030 \end{pmatrix}$$

$$B_p = \begin{pmatrix} 0 & 0.0194 & 0 & 0 \\ -0.0215 & 0 & -0.0040 & -1.7860 \\ 0 & 0 & 0 & 0 \\ -1.0970 & 0 & 0.3660 & -0.0569 \\ 0 & 0 & 0 & 0 \\ -0.6400 & 0 & 0.1625 & -0.0370 \\ 0 & 0 & 0 & 0 \\ -1.8820 & 0 & 0.4720 & -0.0145 \end{pmatrix}$$

$$C_p = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$D_p = 0$$

We consider a fourth-order model by considering only the dominant state variables, i.e., in terms of $\boldsymbol{x}_p = (v, \alpha, \theta, q)^*$. The reduced-order model has the form

$$\dot{\boldsymbol{x}}_m = A_m \boldsymbol{x}_m + B_m \boldsymbol{u}, \qquad \boldsymbol{x}_m(0) = 0, \qquad \boldsymbol{y}_m = C_m \boldsymbol{x}_m \tag{60}$$

where the initial values of various matrices in Eq. (60) were obtained by truncation of the plant model matrices corresponding to the preceding dominant state variables.

For the optimization problem in Eq. (52), we consider the output weighting matrix $Q = \mathrm{diag}(0.5, 0.1, 0.1, 0.1)$, and the control weighting matrix $R = \mathrm{diag}(0.1, 0.1, 0.1, 0.1)$. The time horizon for optimization is taken as $[0, 5]$. Simulations were carried out following the method outlined earlier. The various matrices for the optimized reduced-order model are as follows:

$$A_m = \begin{pmatrix} -0.0199 & -0.0920 & -0.0200 & 0.0213 \\ -0.0974 & -0.1969 & -0.1254 & 0.8369 \\ -0.0045 & -0.0068 & -0.0135 & 1.0053 \\ -0.2306 & 1.8503 & -0.0874 & -0.7980 \end{pmatrix}$$

$$B_m = \begin{pmatrix} 0.0395 & 0.0187 & -0.0158 & 0.0770 \\ 0.2379 & -0.0008 & -0.0632 & -1.7301 \\ 0.0812 & -0.0007 & -0.0253 & 0.0737 \\ -0.9066 & -0.0008 & 0.3194 & 0.0057 \end{pmatrix}$$

$$C_m = \begin{pmatrix} 0.9971 & 0.0062 & 0.0274 & 0.0091 \\ 0.0060 & 0.9985 & -0.0848 & -0.0175 \\ 0.0080 & 0.0066 & 0.8954 & -0.0027 \\ 0.0067 & 0.0049 & -0.0915 & 0.9923 \end{pmatrix}$$

$$D_m = 0$$

The value of the correlation index ρ increased from 0.0029 to 0.8658. Excellent correlation between the original plant response and the reduced-order model response was obtained for step commands applied at the elevator, throttle, canard, and elevon inputs. Figures 2–4 show some of the plots. The rest of them are omitted for the sake of brevity.

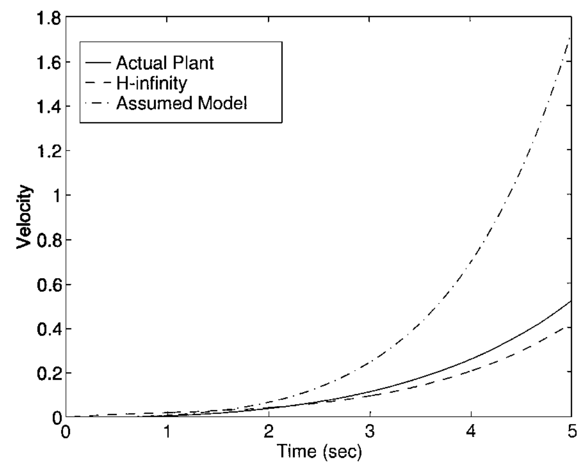

Fig. 2 Velocity due to application of an elevator step.

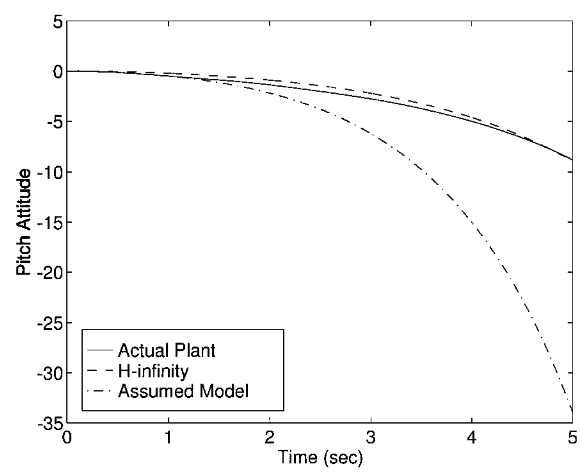

Fig. 3 Pitch attitude due to application of an elevator step.

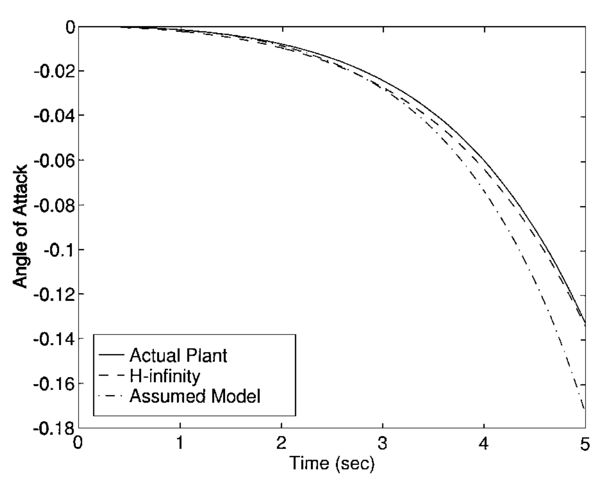

Fig. 4 Angle of attack due to application of a throttle step.

The eigenvalues of the full-order model are given by $-0.0150 \pm 0.0886i$ (phugoid), -1.7756, 0.6687 (short period), $-0.3122 \pm 4.4485i$ (first elastic mode), and $-0.7257 \pm 6.7018i$ (second elastic mode). The reduced-order model eigenvalues are -1.7944, 0.5957 (short period), -0.0097 (phugoid), and 0.1802. Note that in the reduced-order case, an additional unstable root at 0.1802 is introduced.

V. Conclusions

We presented a technique for reduced-order modeling using a finite horizon H_∞ optimality criterion. A characterization for the determination of the correlation between plant and model outputs is given. Algorithms are developed for computing the correlation index and the gradients of the correlation index with respect to reduced-order model parameters. The methodology is utilized in the computation of reduced-order models in two examples.

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